Korpi's Last-Minute AP Things Not to Forget to Remember

- Integration by u-substitution
  - When your "inside" function is linear, and it's derivative is "off" by more than a constant.
  - Let $u =$ "inside" function and change all variables from x, dx to u, du.
  - **Example**: $\int_0^1 \frac{2x}{\sqrt{x+2}}dx$ ⇒ "off by x"

$u = x+2$

$\frac{du}{dx} = 1$

$du = dx$

$x = u-2$

$\int_2^3 \frac{2(u-2)}{\sqrt{u}}du$

$\int_2^3 (2u-4)u^{-1/2}du$

$\int_2^3 (2u^{1/2} - 4u^{-1/2})du$

$(2)(\frac{2}{3})u^{3/2} - (4)(2)u^{1/2}\Big|_2^3$

$\left[\frac{4}{3}(3)^{3/2} - 8(3)^{1/2}\right] - \left[\frac{4}{3}(2^{3/2}) - 8(2^{1/2})\right]$

or $\int_1^4 x\sqrt{3x^2-1}\,dx$

$u = 3x^2-1$

$\frac{du}{dx} = 6x$

$du = 6x\,dx$

$x\,dx = \frac{1}{6}du$

$\boxed{\frac{1}{6}\int_2^{47}\sqrt{u}\,du}$

- Logarithmic Differentiation (LOG DIFF)
  - When you are taking the derivative of a variable function raised to a power of a variable function.
  - Take the natural log of both sides, differentiate implicitly, then resolve for y.
  - **Example**: $\frac{d}{dx}\left[x^{\sin x}\right] =$

$y = x^{\sin x}$

$\ln y = \sin x \cdot \ln x$

$\frac{d}{dx}: \frac{1}{y}\cdot\frac{dy}{dx} = \cos x \cdot \ln x + \sin x\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \left[\cos x \cdot \ln x + \sin x\left(\frac{1}{x}\right)\right]\cdot x^{\sin x}$

- Derivatives of Trig/inverse Trig functions
  - $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$
  - $\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$
  - $\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$
  - $\frac{d}{dx}\operatorname{arccot} x = \frac{-1}{1+x^2}$
  - $\frac{d}{dx}\operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}}$
  - $\frac{d}{dx}\sin x = \cos x$
  - $\frac{d}{dx}\tan x = \sec^2 x$
  - $\frac{d}{dx}\sec x = \sec x \tan x$
  - $\frac{d}{dx}\cos x = -\sin x$
  - $\frac{d}{dx}\cot x = -\csc^2 x$
  - $\frac{d}{dx}\csc x = -\csc x \cot x$
  - Example: $\frac{d}{dx}\cos^{-1}\left(\tan\left(2x^2\right)\right) =$

$\frac{-1}{\sqrt{1-(\tan(2x^2))^2}}\cdot \sec^2(2x^2)\cdot 4x$

- Integral of Trig functions
  - $\int \sin x\,dx =$ $-\cos x + C$
    $\int \cos x\,dx =$ $\sin x + C$
    $\int \tan x\,dx =$ $\int \frac{\sin x}{\cos x}dx$ $-\ln|\cos x| + C$
    $\int \cot x\,dx =$ $\ln|\sin x| + C$
  - $\int \sec x\,dx =$ $\ln|\sec x + \tan x| + C$
    $\int \csc x\,dx =$ $-\ln|\csc x + \cot x| + C$
    $\int \sec^2 x\,dx =$ $\tan x + C$
    $\int \sec x \tan x\,dx =$ $\sec x + C$
  - **Examples**: $\int \tan^2 x\,dx =$
    $\int (\sec^2 x - 1)\,dx =$
    $\boxed{\tan x - x + C}$

    PIDs: $\cos^2 x + \sin^2 x = 1$
    $1 + \tan^2 x = \sec^2 x$
    $1 + \cot^2 x = \csc^2 x$

    $\int \frac{\cot(\sqrt{x})}{\sqrt{x}}dx =$

    $x^{1/2} \xrightarrow{\text{deriv}} \frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\frac{1}{\sqrt{x}}\right)$

    $2\ln|\sin\sqrt{x}| + C$

- Finding extrema vs. finding the location of extrema
  - An extreme value is a *y*-value. It occurs at an *x*-value
  - **Example**: Find the maximum value of $f(x) = x^2 - 5$ on $[-1,2]$

    $f'(x) = 2x = 0$
    $x = 0 \in [-1,2]$

    $f(-1) = -4$
    $f(2) = -1$
    $f(0) = -5$

    $\boxed{\text{So } f \text{ has a max of } -1}$

- Finding slopes of inverse functions
  - Inverse functions, at corresponding points, have reciprocal slopes.
  - If $f(g(x)) = x = g(f(x))$, then $f(x)$ and $g(x)$ are inverses
  - $g(a) = b$ implies $f(b) = a$
  - $g'(a) = \dfrac{1}{f'(b)}$
  - **Example**: If $f(x) = g^{-1}(x)$ and $f(x) = 2x^2 + 3x - 1$ and if $g(-2) = -1$, find $g'(-2)$.

    $f'(x) = 4x + 3$
    $f'(-1) = -4 + 3$
    $f'(-1) = -1$

    $g: (-2,-1)$
    $f: (-1,-2)$

    So $g'(-2) = \frac{1}{f'(-1)}$
    $= \frac{1}{-1}$
    $= \boxed{-1}$

- Finding the slope of a normal line to a function, $f(x)$, at a point $x = a$
  - Normal lines are perpendicular to tangent lines at a point.
  - The normal slope, n, is the opposite, reciprocal of the tangent slope.
  - $n = \dfrac{-1}{f'(a)}$
  - **Example**: Find the equation of the normal line to the graph of $y = e^{2x}$ at $x = \ln 2$

$y' = e^{2x} \cdot 2$

$y'(\ln 2) = 4 \cdot 2 = 8 = m$

$\boxed{n = -\frac{1}{8}}$

$(x, y) = (\ln 2, e^{2(\ln 2)})$

$(\ln 2, e^{\ln 2^2})$

$\boxed{(\ln 2, 4)}$

eq: $y = 4 - \frac{1}{8}(x - \ln 2)$

- Squiggle Alert when the words "approximate" or "estimate" are used in the question
  - Explicitly stated approximations must have an approximation symbol, $\approx$, rather than an equal sign.
  - This happens with tangent line approximations, numeric methods of integration, linearization, Euler's Method (BC), and Taylor Polynomials (BC).
  - **Example**: If f is differentiable, and if $f(3) = 2$ and $f(5.5) = 5$, approximate $f'(4.1)$. Show the work that leads to your answer.

$f'(4.1) \approx \dfrac{5-2}{5.5-3}$

- Average value vs. average rate of change
  - Average value is the average y-value for whatever is being measured on the y-axis.
  - Average value is "integral over width"
  - Average value $= \dfrac{\int_a^b f(x)\,dx}{b-a} = \dfrac{1}{b-a}\int_a^b f(x)\,dx$
  - Average rate of change is the change in y over the change in x—the slope of the secant line
  - Average rate of change $= \dfrac{f(b) - f(a)}{b-a}$
  - **Example**: If $W(t)$ is the rate at which rain falls on a roof of a house for $0 \le t \le 3$, where W is measured in cm/hr and t is measured in hours. Explain the meaning, with correct units, of $\frac{1}{3}\int_0^3 W(t)\,dt$ in the context of rainfall.

cm/hr

Avg rate of rainfall in cm/hr on roof from t=0 hr to t=3 hr.

- The misuse of equality a.k.a. mathematically prevarication = lie
  - If you use an equal sign, the expressions you are equating better be equal, or you will lose a point.
  - **Example**: If $f(x) = 2x^2 - 1$, find $f'(1)$
    - WRONG: $f'(x) = 4x = 4$
    - CORRECT: $f'(x) = 4x$, $f'(1) = 4(1) = 4$
  - **Example**: To the nearest whole number, find $\int_0^2 e^x \, dx$
    - WRONG: $\int_0^2 e^x \, dx = 6.389 = 6$
    - CORRECT: $\int_0^2 e^x \, dx = 6.389 \approx 6$ OR $\int_0^2 e^x \, dx = 6$

- Be sure to include units in all final numeric answers AND any written explanation of this answer (including both independent AND dependent variables.
  - Units can cost you an entire point if you omit them or use the wrong ones.
  - **Example**: $w(t)$ is the temperature of water in a jug in a refrigerator, in $^\circ F$, where t is in minutes.

    In the context of the problem, explain the meaning of (a) $w'(5) = -2.1$ (b) $\frac{1}{5}\int_0^5 w(t)\,dt = 44$

    (a) At $t=5$ min, the water's temp is decreasing by 2.1 °F per min.

    (b) from $t=0$ min to $t=5$ min, the Avg temp of water is 44°F

- Speed increasing or decreasing vs. Velocity increasing or decreasing
  - If at $t = c$, $v(c) > 0$ AND $a(c) > 0$ or $v(c) < 0$ AND $a(c) < 0$, then speed is increasing at $t = c$.
  - If at $t = c$, $v(c) < 0$ AND $a(c) > 0$ or $v(c) > 0$ AND $a(c) < 0$, then speed is decreasing at $t = c$.
  - If at $t = c$, $v'(c) = a(c) > 0$, then velocity is increasing at $t = c$.
  - If at $t = c$, $v'(c) = a(c) < 0$, then velocity is decreasing at $t = c$.
  - If the graph of $v(t)$ moves TOWARD the t axis, speed is decreasing.
  - If the graph of $v(t)$ moves AWAY FROM the t axis, speed is increasing.
  - **Example**: If a particle moves along the x-axis such that for $t \ge 0$, its position is give by ,

    $x(t) = \frac{1}{3}t^3 - 4t^2 + 15t - 7$, at $t = 4.5$, is the speed of the particle increasing or decreasing? At this time is the velocity of the particle increasing or decreasing? Justify your answers.

    $x'(t) = v(t) = t^2 - 8t + 15$, $v(4.5) = -0.75 < 0$

    $v'(t) = a(t) = 2t - 8$ $\qquad a(4.5) = 1 > 0$

    So at $t=4.5$, the speed is decreasing.

    But, since $v'(4.5) = a(4.5) = 1 > 0$, at $t=4.5$, velocity is increasing.

- IVT (The Intermediate Value Theorem)
  - If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ takes on all y-values between $f(a)$ and $f(b)$.
  - **Example**: If $f(x)$ is a differentiable function such that $f(-1)=-3$ and $f(4)=\frac{5}{6}$, explain why $f(x)$ must have a root on the interval $(-1,4)$. $f(x)$ is differentiable, so $f(x)$ is continuous since f has a root when $f(c)=0$ for some $x=c$, and since $f(-1)=-3<0<\frac{5}{6}=f(4)$, by the IVT, f must have a zero on $(-1,4)$.

- EVT (The Extreme Value Theorem)
  - If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ has both a Maximum and Minimum on the CLOSED interval $[a,b]$.
  - **Example**: If a particle moves along the x-axis such that its position is give by, $x(t)=t^2-3t+2$, on the interval $0 \le t \le 2$, at what time, t, is the particle farthest left? Farthest right? $x(t)$ is continuous $\forall x \in \mathbb{R}$

    Minimum position Max position

    $x'(t)=2t-3=0$

    $\boxed{t=\frac{3}{2}}$ critical value

    endpts $\begin{cases} x(0)=2 \leftarrow \text{MAX} \\ x(2)=0 \end{cases}$

    c.v. $\{ x(\frac{3}{2})=\frac{9}{4}-\frac{9}{2}+2=-\frac{1}{4} \leftarrow$ MIN

    So particle is farthest left at $t=3/2$ & farthest right at $t=0$.

- MVT (The Mean Value Theorem)
  - If $f(x)$ is continuous on a closed interval $[a,b]$, and differentiable on the open interval (a,b), then there is an $x=c$ on the OPEN interval (a,b), where the slope of the tangent line (instantaneous rate of change/derivative) equals the slope of the secant line (average rate of change).
  - USED TO SHOW THAT A DERIVATIVE EXISTS ON AN INTERVAL
  - Set it up as $f'(x)=\dfrac{f(b)-f(a)}{b-a}$, then solve for x, then make sure x is in the OPEN interval!
  - **Example**: If $f(x)=x^3+x-4$, on the interval $-1 \le x \le 2$, find the value of c guaranteed by the Mean Value Theorem.

    $f'(x)=\frac{f(2)-f(-1)}{2-(-1)}$

    $3x^2+1=\frac{6+6}{3}=4$

    $\left.\begin{array}{l} 3x^2+1=4 \\ 3x^2=3 \end{array}\right\}$ $x^2=1$, $x=-1$, $\boxed{x=1}$ only since $x=-1 \notin (-1,2)$ open interval
  - **Example:** If $f'(x)$ is a differentiable function for all x, and if $f'(5)=-2$ and $f'(7)=4$, explain why there must be a c, $5<c<7$ such that $f''(c)=3$. Since f' is differentiable, it is continuous by the MVT: $f''(c)=\frac{f'(7)-f'(5)}{7-5}=\frac{4+2}{2}=3$ for some $5<c<7$.

- Geometric formulas to remember
  - Volume of a Sphere: $V = \frac{4}{3}\pi r^3$ Surface area of a Sphere: $A = 4\pi r^2$
  - Volume of a Cone: $V = \frac{\pi}{3}r^2 h$ Volume of a Cylinder: $V = \pi r^2 h$
  - Surface area of a Cylinder: $A = 2\pi r^2 + 2\pi rh$ Equilateral Triangle: $A = \boxed{\frac{\sqrt{3}}{4}}s^2$
  - Trapezoid: $A = \boxed{\frac{1}{2}\Delta x\left(y_1 + y_2\right)}$ Rectangle: $A = h \cdot w$

- Justifying relative extrema using the First Derivative test and Second Derivative Test
  - First Derivative Test (at a critical point, $\left(c, f\left(c\right)\right)$)
    - "Since $f'\left(c\right) = 0$ (or $f'\left(c\right) = DNE$), and since $f'\left(x\right)$ changes from positive to negative at $x = c$, $f\left(x\right)$ has a Relative (local) Maximum at $x = c$."
    - "Since $f'\left(c\right) = 0$ (or $f'\left(c\right) = DNE$), and since $f'\left(x\right)$ changes from negative to positive at $x = c$, $f\left(x\right)$ has a Relative (local) Minimum at $x = c$."
  - Second Derivative Test (at a critical value$\left(c, f\left(c\right)\right)$)
    - "Since $f'\left(c\right) = 0$ (or $f'\left(c\right) = DNE$), and since $f''\left(c\right) < 0$, $f\left(x\right)$ has a Relative (local) Maximum at $x = c$."
    - "Since $f'\left(c\right) = 0$ (or $f'\left(c\right) = DNE$), and since $f''\left(c\right) > 0$, $f\left(x\right)$ has a Relative (local) Minimum at $x = c$."

- Justifying an inflection point at a p.i.v. (possible inflection value)
  - If $f\left(c\right)$ is defined, and either $f''\left(c\right) = 0$ or $f''\left(c\right) = DNE$, then $f\left(x\right)$ has an inflection point at $\left(c, f\left(c\right)\right)$ if f'' changes from positive to negative at $x = c$ or negative to positive at $x = c$.
  - **Example**: A continuous function $f\left(x\right)$ has a second derivative $f''\left(x\right) = \frac{\left|x-3\right|}{x-3}$. Determine if $f\left(x\right)$ has an inflection value or not at $x = 3$. Justify.

$f''(x) = DNE$ @ $x=3$

x	2	3	4
f''	$-$		$+$

Since f'' changes from positive to negative at $x=3$, f has an inflection value @ $x=3$

- Cross-sectional volume magic numbers
  - Squares: 1 Equilateral Triangles: $\frac{\sqrt{3}}{4}$ Semicircles: $\frac{\pi}{8}$ Rectangles with height n times the base n
  - Quarter Circles: $\frac{\pi}{4}$ Isos Rt Triangle, Leg in Base: $\frac{1}{2}$ Isos Rt Triangle, Hypot in Base: $\frac{1}{4}$
- Inverse Trig Integral formulas
  - $\int \frac{du}{a^2+u^2} =$ $\frac{1}{a}\arctan\frac{u}{a}+C$ $\int \frac{du}{\sqrt{a^2-u^2}} =$ $\arcsin\frac{u}{a}+C$ $\int \frac{du}{u\sqrt{u^2-a^2}} =$ $\frac{1}{a}\operatorname{arcsec}\frac{|u|}{a}+C$
  - **Examples**:

$\int \frac{1}{x^2+5}dx =$ $u=x$ $a=\sqrt{5}$ $\frac{1}{\sqrt{5}}\arctan\frac{x}{\sqrt{5}}+C$

$\int \frac{5}{\sqrt{7-9x^2}}dx =$ $a=\sqrt{7}$ $u=3x$ $(5)\left(\frac{1}{3}\right)\arcsin\frac{3x}{\sqrt{7}}+C$

$\int \frac{1 e^x}{e^x\sqrt{e^{2x}-2}}dx =$ $u=e^x$ $a=\sqrt{2}$ $\frac{1}{\sqrt{2}}\operatorname{arcsec}\frac{|e^x|}{\sqrt{2}}+C$ $\frac{1}{\sqrt{2}}\operatorname{arcsec}\frac{e^x}{\sqrt{2}}+C$

BC

- Convergence Tests (used also to determine endpoints of intervals of convergence)
  - nth term test for divergence
    $\sum a_n$ if $\lim_{n\to\infty} a_n \neq 0$, Series diverges
  - Geometric series
    $\sum a\cdot r^n$, converges to $\frac{\text{1st term}}{1-r}$ if $|r|<1$ otherwise, diverges
  - p-series
    $\sum \frac{1}{n^p}$ converges if $p>1$ Diverges if $p\le 1$
  - Direct/Limit Comparison Test
    Compare to a know series
  - Integral Test
    if $\int_1^\infty f(x)dx$ converges, or Diverges $\sum_{n=1}^\infty a_n$ Converges too or Diverges
  - Ratio Test $\sum a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$ diverges if >1
  - Alternating Series Test/Error
    $\sum (-1)^n a_n$ Converges if $\lim_{n\to\infty} a_n = 0$
- Pen or Pencil on the exam???
  Black or Dark Blue/Blue (No Lavender w/sparkles)
  ~~erase~~
- 3-decimal accuracy (round or truncate). Store non-exact answers needed for future calculations and LABEL THEM ON YOUR PAPER. Avoid duplicate letters. NEVER use an approximate answer to calculate a subsequent value.

- Implicit Differentiation—your derivative will have both x and y in it.
  - You will have as many $\frac{dy}{dx}$'s in your derivative as you have y's in your equation.
  - Look to solve for y first, especially if your answer choices are in terms of x only and/or you are finding an actual value and are only given $x = a$.
  - When solving for $\frac{dy}{dx}$, if you ever end up with an answer like $\frac{dy}{dx} = \frac{a-b}{c-d}$, realize that this is equivalent to $\frac{dy}{dx} = \frac{b-a}{d-c}$.
  - When finding a second derivative (or higher order derivative) implicitly, if the instructions say "in terms of x and y," be sure to plug in your $\frac{dy}{dx}$ expression into your final answer.

- Parenthesis are your best friends—it's better to have them and not need them than to need them and not have them. This is especially true for:
  - $k\int_a^b f(x)\,dx = k\left[\left(f(b)\right) - \left(f(a)\right)\right]$ $\qquad$ $\int_a^b \left[f(x) + g(x)\right]dx$
  - $\pi\int_a^b \left[\left(R(x)\right)^2 - \left(r(x)\right)^2\right]dx$ $\qquad$ $\lim\limits_{h\to 0} \frac{\left((x+h)^2 - 2(x+h) + 1\right) - \left(x^2 - 2x + 1\right)}{h} =$
  - $5 - \cos^2 x = 5 - \left(1 - \sin^2 x\right)$ $\qquad$ $\frac{5}{2-\sqrt{x-3}} \cdot \frac{2+\sqrt{x-3}}{2+\sqrt{x-3}} = \frac{5\left(2+\sqrt{x-3}\right)}{4-(x-3)}$

- Cusp Alert!
  - If you have a variable raised to a power that is between zero and one, you will have a continuous function that is not differentiable. **The root of the term with the alerted power will be a critical value of the function!**
  - **Example**: Find the critical values of $f(x) = x^{2/3} - x$

$f(x) = (x-3)^{4/5}$

cusp/c.v. @ $x = 3$

c.v. at root of factor w/ exponent

C.V. $x = 0$

$f'(x) = \frac{2}{3}x^{-1/3} - 1 = 0$

$\frac{2}{3\sqrt[3]{x}} = 1$

$3\sqrt[3]{x} = 2$

$\sqrt[3]{x} = \frac{2}{3}$

$\boxed{x = \frac{8}{27}}$

critical value

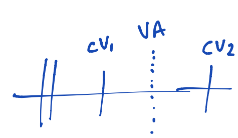
$\boxed{\text{horz tangent}}$

$f'(x) = \text{DNE}$

$\boxed{x = 0}$

* In any number line chart c.v.s / p.i.v.s & any discons (VA)

1st deriv $\quad$ 2nd deriv



- Pronouns. Don't Use Pronouns. Don't be vague, ambiguous, or unclear either.
    - Don't say things like, " . . . since **<u>it</u>** changes from positive to negative . . .," "since **<u>the graph</u>** is increasing," or "**<u>the function</u>** changes signs **<u>there</u>**."
    - Be explicit. Say what you mean and mean what you say.

- Don't waste time erasing. If you draw a line through something, it becomes "invisible" to the AP graders. ~~Why So Serious?~~, ~~We Are Sparta.~~

- Don't draw a line through anything on a free response or erase anything unless you have the time and intention of replacing it with something else. Something is better than nothing. Never leave anything blank, whether it's a M.C. or F.R. question. You EARN points on this test, not lose points.

- Don't let a wrong answer on one part of a F.R. question keep you from getting credit on subsequent parts. Using your wrong answer correctly, making up a reasonable equation to work with, or even attaching units to a number can get you points. NEVER give up. NEVER surrender.

- You don't need to simplify your numeric answers on the free response (unless the instructions explicitly tell you to approximate it to 3 decimals or ask you to "show" that a number equals a given number), but you MUST indicate your numeric methods to get credit. This especially goes for integral approximations from a table of values, difference quotients, and using areas from a graph to approximate integrals.

- Breathe, Relax, Smile, and get that 5!